Outcome 3 – Solving quartic equations

Worked Example:

Given that $z = 1 + 3i$ is a root of the equation $z^4 - 8z^3 + 32z^2 - 80z + 100 = 0$,

(a) Write down another root of this equation
(b) Find all the roots of the equation

Key Facts/Formulae:

We can find roots using;

- Synthetic division
- The quadratic formula
- The conjugate roots property

We can find factors by dividing!

Essential knowledge!

Complex roots of a polynomial equation (with real coefficients) occur in conjugate pairs.

E.g. if $a + bi$ is a root, then so is $a - bi$.

1. Write down another root

$z = 1 + 3i$

2. Form two factors

If $z = 1 + 3i$ is a root then $z - (1 + 3i)$ is a factor.

If $z = 1 - 3i$ is a root then $z - (1 - 3i)$ is a factor.

2. Multiply the two factors to form a trinomial

$$\{(z-1) - 3i\}\{(z-1) + 3i\} = z^2 - 2z + 1 + 3\cancel{i} - \cancel{3i} - 3\cancel{i} + \cancel{3i} + 9 = z^2 - 2z + 10$$

3. Use algebraic long division to find the other factor

$$\begin{array}{r|l} & z^2 - 6z + 10 \\ \hline z^2 - 2z + 10 & z^4 - 8z^3 + 32z^2 - 80z + 100 \\ & -\ z^4 - 2z^3 + 10z^2 \\ \hline & \quad -6z^3 + 22z^2 - 80z \\ & -\ -6z^3 + 12z^2 - 60z \\ \hline & \quad 10z^2 - 20z + 100 \\ & -\ 10z^2 - 20z + 100 \\ \hline & \quad 0 \end{array}$$

4. Factorise/Quadratic formula to find the remaining root(s)

$$z = \frac{6 \pm \sqrt{36 - 4(10)}}{2} = \frac{6 \pm \sqrt{-4}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$$

$z = 1 + 3i$ $z = 1 - 3i$ $z = 3 + i$ $z = 3 - i$

Questions…

1. Given that $z = 1 + i$ is a root of the equation $z^4 - 3z^3 - 2z^2 + 10z - 12 = 0$,

   (a) Write down another root of this equation
   (b) Find all the roots of the equation

2. Given that $z = 2 + 3i$ is a root of the equation $z^4 + z^3 - 3z^2 + 49z + 52 = 0$,

   (a) Write down another root of this equation
   (b) Find all the roots of the equation

3. Given that $z = -2 - 2i$ is a root of the equation $z^4 - 6z^3 - 7z^2 + 20z + 200 = 0$,

   (a) Write down another root of this equation
   (b) Find all the roots of the equation

4. Given that $z = -1 - i$ is a root of the equation $z^4 + 22z^3 + 142z^2 + 240z + 200 = 0$,

   (a) Write down another root of this equation
   (b) Find all the roots of the equation

5. Given that $z = 1 + 5i$ is a root of the equation $z^4 - 12z^3 + 62z^2 - 312z + 676 = 0$,

   (a) Write down another root of this equation
   (b) Find all the roots of the equation

6. Given that $z = -3 + 2i$ is a root of the equation $z^4 + 2z^3 + 9z^2 + 68z + 260 = 0$,

   (a) Write down another root of this equation
   (b) Find all the roots of the equation

Answers

1 $z = 1 + i$ $z = 1 - i$ $z = 3$ $z = -2$

2 $z = 2 + 3i$ $z = 2 - 3i$ $z = -1$ $z = -4$

3 $z = -2 - 2i$ $z = -2 + 2i$ $z = 5$

4 $z = -1 - i$ $z = -1 + i$ $z = -10$

5 $z = 1 + 5i$ $z = 1 - 5i$ $z = 5 + i$ $z = 5 - i$

6 $z = -3 + 2i$ $z = -3 - 2i$ $z = 2 + 4i$ $z = 2 - 4i$