Outcome 4 - The Chain Rule, Product Rule and Quotient Rule with further trigonometric functions

Worked Example:

Show that the derivative of $\sec x$ is $\sec x \tan x$.

1. Define function and get in a differentiable form.

Let $y = \sec x = \dfrac{1}{\cos x} = (\cos x)^{-1}$

2. Choose strategy and differentiate. (Chain Rule)

$$\frac{dy}{dx} = -(\cos x)^{-2} \times (-\sin x) = \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \sec x \tan x \text{ as required.}$$

Key Facts/Formulae:

The Chain Rule — f′(outside) × f′(inside)

If $y = u$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

The Product Rule

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

The Quotient Rule

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

NOT given on formula sheet!

Essential prior knowledge!

$f(x) = \sin ax$
$f'(x) = a\cos ax$

$f(x) = \cos ax$
$f'(x) = -a\sin ax$

Advanced Higher Formula sheet

$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\cot x$	$-\text{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Questions...

Prove each of the following;

1. Show that the derivative of $\tan x$ is $\sec^2 x$.
2. Show that the derivative of $\cot x$ is $-\text{cosec}^2 x$.
3. Show that the derivative of $\sec x \cos^2 x$ is $-\sin x$.
4. Show that the derivative of $\sec^2 x \cos^2 x$ is 0.
5. Show that the derivative of $\tan^2 x + 1$ is $2\tan x \sec^2 x$.
6. Show that the derivative of $\text{cosec}\, x + \cot x$ is $-\dfrac{1 + \cos x}{\sin^2 x}$.

Answers

1 $y = \tan x = \dfrac{\sin x}{\cos x}$ Quotient rule!

$$\frac{dy}{dx} = \frac{\cos^2 x - \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

2 $y = \cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$ Quotient rule!

$$\frac{dy}{dx} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{\cos^2 x + \sin^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\mathrm{cosec}^2 x$$

3 $y = \sec x \cos^2 x = \dfrac{1}{\cos x} \times \cos^2 x = \dfrac{\cos^2 x}{\cos x} = \cos x$

$\dfrac{dy}{dx} = -\sin x$ Basic differentiation!

4 $y = \sec^2 x \cos^2 x = \dfrac{1}{\cos^2 x} \times \cos^2 x = \dfrac{\cos^2 x}{\cos^2 x} = 1$

$\dfrac{dy}{dx} = 0$ Basic differentiation!

5 $y = \tan^2 x + 1 = \dfrac{\sin^2 x}{\cos^2 x} + \dfrac{\cos^2 x}{\cos^2 x} = \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = (\cos x)^{-2}$

$\dfrac{dy}{dx} = -2(\cos x)^{-3} \times (-\sin x) = 2 \times \dfrac{\sin x}{\cos x} \times \dfrac{1}{\cos^2 x} = 2\tan x \sec^2 x$ Chain rule!

6 $y = \mathrm{cosec}\, x + \cot x = \dfrac{1}{\sin x} + \dfrac{1}{\tan x} = \dfrac{1}{\sin x} + \dfrac{\cos x}{\sin x} = \dfrac{1 + \cos x}{\sin x}$ Quotient rule!

$$\frac{dy}{dx} = \frac{-\sin^2 x - \cos x\,(1 + \cos x)}{\sin^2 x} = \frac{-\sin^2 x - \cos x - \cos^2 x}{\sin^2 x} = -\frac{\cos^2 x + \sin^2 x + \cos x}{\sin^2 x} = -\frac{1 + \cos x}{\sin^2 x}$$